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IMPULSIVE LOADING OF RIGID VISCOPLASTIC PLATES*

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Abstract—The response of thin simply supported circular metal plates to impulsive loading is investigated. The constitutive equations employed are based on the Huber–Mises yield condition and account for the effect of strain rate sensitivity of the material. The hypothesis concerning proportional loading for viscoplastic material has been used to linearize the system of governing equations. Explicit formulas have been obtained for the plate velocity and permanent plastic deflection under various conditions of plate loading. Comparison with recent experimental data and with the known perfectly plastic solution is presented. It is found that the shape of the deformed plate predicted by the theory always conforms to the observed deflection. By appropriate choice of the material constants a good degree of agreement between theory and experiments can be achieved within a certain interval of the applied impulse.

1. INTRODUCTION

ANALYSIS of the complete response of strain rate sensitive structures to dynamical loading has so far been confined almost exclusively to one-dimensional problems including bending of beams and impact of rods and cylinders [1, 2]. It was found that the domain in which a nonlinear parabolic equation was to be solved contains one moving and unknown boundary. The determination of this boundary separating regions of viscoplastic and rigid behavior constitutes the major mathematical difficulty in the analytical treatment of the problem.

On the other hand, for some plate problems, the biaxial state of stress demands that the entire plate be in the viscoplastic range, thus the domain of integration has constant boundaries which simplifies the mathematics involved. Therefore, the dynamic response of rigid viscoplastic plates, governed by a quasi-linear system of parabolic equations is an attractive example for studying the influence of the rate of strain on the behavior of structures in combined states of stress.

In the author's previous paper [3], the general method of numerical solution for any circular or annular plate was presented in the case of the Huber–Mises yield condition and a power-type of function describing the viscous properties of material. Results concerning a simply supported circular plate loaded by uniformly distributed pressure have indicated large discrepancies between that kind of solution and the prediction of the simple rigid perfectly plastic theory. The differences concern both the shape of the deformed plate and the value of the permanent central deflection. In order to perform a more detailed study of the viscoplastic response of plates and also compare the results obtained with existing solutions and some recent experimental data it is necessary to treat the same problem analytically.

The objective of the present paper is to derive closed form solutions for circular, simply supported plates. To this end the constitutive equations are linearized by introducing the

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hypothesis of proportional loading. The influence of various parameters such as dimensions of the plate, constants of materials, including the viscosity constant, and conditions of loading will be discussed at length. The present solution will be compared with recent experimental results by Florence [4] and with the solutions of Hopkins and Prager [5] and Wang [6], based upon the perfectly plastic material.

2. CONCEPT OF PROPORTIONAL LOADING AND LINEARIZATION OF BASIC EQUATION

The problem cannot be treated analytically unless the governing system of quasi-linear parabolic equations is linearized*. In the present paper the linearization of the governing equations is achieved using a concept of proportional loading introduced by Baltov [7] and described in one particular case by Plass [8]. Consider an incompressible rigid, viscoplastic material and following Perzyna [9] assume that the strain rate is a function of the excess of the current state of stress above the initial yield surface

$$\dot{\varepsilon}_{ij} = \gamma_0 \varphi(F) \frac{\partial F}{\partial \sigma_{ij}}, \tag{2.1}$$

where $F = (\sqrt{J_2}/k) - 1$ and J_2 and k are the second invariant of the stress deviator and yield stress in simple shear, respectively. Denote by $F = 0$ the initial yield surface in the space of deviators and by $\bar{s}_{ij}$ the state of stress on this surface. According to (2.1) the strain rate for $F \leq 0$ is identically zero. Let $F_1, F_2, \ldots$ be subsequent yield surfaces for non zero strain rate.

Proportional loading requires the direction cosine tensor to be independent of time, $s_{ij}/\sqrt{J_2} = \bar{s}_{ij}/k$. For rigid, perfectly plastic bodies, where $\sqrt{J_2} = k$, proportional loading means the constant value of stresses throughout the whole deformation process. For viscoplastic materials the condition of proportional loading is fulfilled if the stress trajectory is a straight line passing through the origin in the space of s_{ij}. If we denote by $\bar{s}_{ij}$ the state of stress on the surface $F = 0$, then for proportional loading, equation (2.1) is equivalent to the following

$$\dot{\varepsilon}_{ij} = \gamma \varphi\left(\frac{s_{ij}}{\bar{s}_{ij}} - 1\right) \bar{s}_{ij}, \tag{2.2}$$

where viscosity constant $\gamma = \gamma_0/k$. Equation (2.2) turns into a linear relation between components of stress and strain rate if $\varphi(F) = F$, whereas equation (2.1) remains nonlinear.

For a simply supported circular plate the radial and circumferential moments M_r and M_φ are equal to each other at the center of the plate $r = 0$, and the radial moment vanishes on a simply supported edge $r = R$. Therefore for two extreme points of the plate the moment trajectories are straight lines and this suggests that a state close to proportional loading is realized everywhere in the plate. To justify this hypothesis, moment trajectories for a few characteristic values of the plate radius have been plotted on Fig. 1. These are the numerical results obtained in the paper by Wierzbicki [3] and concern a rectangular

* The method of linearization in viscoplasticity developed by Prager [10] does not introduce much simplification in the case of dynamic loading. Although the equations become linear, one has to consider several regions with unknown and time variable boundaries. The continuity conditions to be satisfied on these boundaries are non-linear and hence the whole initial-boundary value problem remains nonlinear. An attempt along these lines concerning impulsive loading of a circular plate is due to Gizatulina [11].

pulse of uniformly distributed pressure. The right hand side of each loop, for instance AB, corresponds to the loading (application of pressure) whereas the left hand side BCD, to the unloading (release of pressure). It is seen that the loops are fairly narrow within a central zone of the plate. One can also expect a compensation effect by the choice of intermediate straight trajectories (broken lines on Fig. 1).

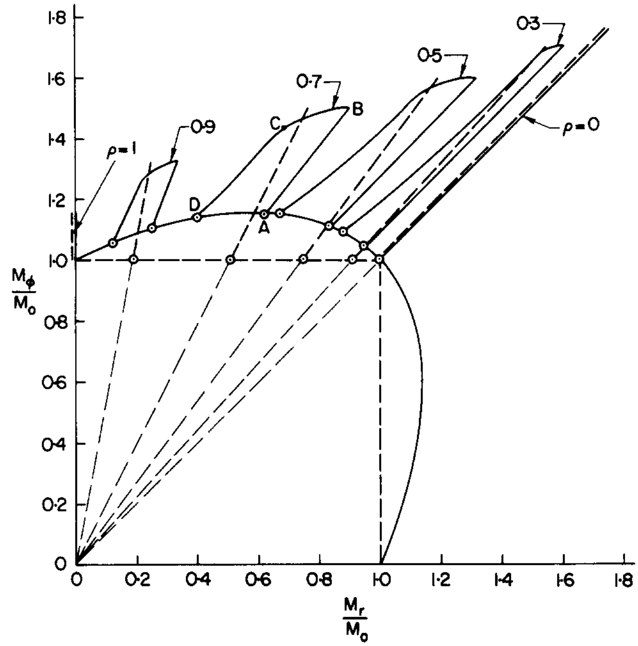

FIG. 1. Moment trajectories for a simply supported circular plate with Huber–Mises yield conditions.

In the case of thin plates with circular symmetry the state of stress is essentially plane. Hence, for the linear function $\varphi(F)$ the constitutive equations (2.1) transformed to the space of generalized stress and strain rates have a form

$$\dot{\kappa}_r = B\left(1-\frac{M_0}{\sqrt{(M_r^2-M_rM_\varphi+M_\varphi^2)}}\right)\frac{2M_r-M_\varphi}{M_0}$$
$$\dot{\kappa}_\varphi = B\left(1-\frac{M_0}{\sqrt{(M_r^2-M_rM_\varphi+M_\varphi^2)}}\right)\frac{2M_\varphi-M_r}{M_0} \tag{2.3}$$

where $B = (\sqrt{3})\gamma/2h$ and h denotes the half thickness of the plate. Assuming proportional loading, equations (2.3), according to (2.2), become

$$\dot{\kappa}_r = \frac{B}{M_0}[(2M_r-M_\varphi)-(2\overline{M}_r-\overline{M}_\varphi)],$$
$$\dot{\kappa}_\varphi = \frac{B}{M_0}[(2M_\varphi-M_r)-(2\overline{M}_\varphi-\overline{M}_r)], \tag{2.4}$$

where $\overline{M}_r(r)$ and $\overline{M}_\varphi(r)$ are moments satisfying for any r the equation of the initial yield surface $M_r^2 - M_r M_\varphi + M_\varphi^2 = M_0^2$.

3. GOVERNING EQUATIONS, BOUNDARY AND INITIAL CONDITIONS

It is convenient to carry out the analysis in dimensionless quantities, defined as follows

$$
\begin{aligned}
m &= \frac{M_r}{M_0}, \qquad n = \frac{M_\varphi}{M_0}, \qquad q = \frac{RQ}{M_0}, \qquad \rho = \frac{r}{R}, \\
v &= \frac{1}{BR^2}\frac{\partial w}{\partial t}, \qquad p'(\rho, t) = p(r, t)\frac{R^2}{M_0}, \qquad M_0 = \sigma_0 h^2
\end{aligned}
\tag{3.1}
$$

where σ_0 is the yield stress in pure tension and Q denotes the shearing force. According to the assumption concerning small deflections of the plate, rate of curvatures $\dot{\kappa}_r$ and $\dot{\kappa}_\varphi$ are related to the rate of deflection $\dot{w}$ by

$$
\dot{\kappa}_r = -\frac{\partial^2 \dot{w}}{\partial \rho^2}; \qquad \dot{\kappa}_\varphi = -\frac{1}{\rho}\frac{\partial \dot{w}}{\partial \rho}. \tag{3.2}
$$

Neglecting the rotary inertia and taking into account the transverse inertia terms the equations of motion can be expressed in the form

$$
\frac{\partial}{\partial \rho}(\rho q) + \rho p'(\rho, t) = \alpha \rho \frac{\partial v}{\partial t}, \qquad \frac{\partial}{\partial \rho}(\rho m) - n = \rho q, \tag{3.3}
$$

where $\alpha = BR^2\mu(R^2/M_0)$ and μ denotes the mass density per unit area of the plate middle surface. Equations (3.2), (3.3) together with (2.4) furnish a linear parabolic system with six unknown functions.

By eliminating all unknowns except v, the system of governing equations can be reduced to

$$
\left[\tfrac{2}{3}\nabla^4 v + \alpha\frac{\partial v}{\partial t} - p\right]\rho = \frac{\partial}{\partial \rho}\left[\frac{\partial}{\partial \rho}(\rho \bar{m}) - \bar{n}\right], \tag{3.4}
$$

where the operator ∇^2 denotes $\nabla^2 = (\partial^2/\partial\rho^2) + (1/\rho \,.\, \partial/\partial\rho)$. Using the equation of equilibrium of the corresponding static problem the right hand side of equation (3.4) can be replaced by $\rho p_0'$ where p_0' denotes the static load-carrying capacity of the plate under similar conditions of loading. It is known that for the Huber–Mises yield condition this value is $p_0' = 6{\cdot}51$. Hence the final form of equation (3.4) becomes

$$
\nabla^4 v + \tfrac{3}{2}\alpha\frac{\partial v}{\partial t} = \tfrac{3}{2}(p' - p_0'). \tag{3.5}
$$

Equation (3.5) is of the same structure as the biharmonic equation describing the forced vibrations of thin elastic plates, but in the latter case in place of velocity stands the plate deflection itself.

It should be stressed here that equation (3.5) has been derived without the knowledge of the static moment distribution $\bar{m}(\rho)$ and $\bar{n}(\rho)$. This property of the proposed method is of particular importance since the explicit formulas for the static moment distribution in the case of Huber–Mises yield condition is not available in the literature.

The boundary conditions for a simply supported plate demand $m = n$ and $q = 0$ at the center of the plate and $v = m = 0$ at the edge. These conditions, expressed in terms of rate of deflection, are

$$\lim_{\rho\to 0}\left(\frac{\partial^2 v}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial v}{\partial\rho}\right) = 0; \qquad \lim_{\rho\to 0}\left(\frac{\partial^3 v}{\partial\rho^3}+\frac{1}{\rho}\frac{\partial^2 v}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial v}{\partial\rho}\right) = 0,$$
$$2\frac{\partial^2 v}{\partial\rho^2}+\frac{\partial v}{\partial\rho}\bigg|_{\rho=1} = 0; \qquad v(1,t).= 0. \tag{3.6}$$

In the general case of plate loading it is assumed that initially the plate is flat and the velocity of all its points is zero

$$w(\rho,0) = 0; \qquad v(\rho,0) = 0. \tag{3.7}$$

The limiting case of impulsive loading will also be considered. Then the initial deflection is zero, whereas the velocity assumes a constant value throughout the entire plate

$$w(\rho,0) = 0, \qquad V(\rho,0) = \frac{I}{\mu}.$$

4. SOLUTION OF THE PROBLEM FOR A STEP TYPE OF PRESSURE LOADING

An initial-boundary value problem for equation (3.5) can be solved by means of an eigenvalue expansion method if the source term $p'(\rho,t)-p'_0$ does not change with time. Let us assume that a suddenly applied pressure is uniformly distributed over the plate surface. Omitting the details of computation the final solution expressed in terms of Bessel and exponential functions is given by*

$$v^I(\rho,t) = \frac{p'-p'_0}{256}\left\{(6\rho^4-28\rho^2+22)-\sum_{n=1}^{\infty}\frac{256}{\lambda_n^5}\psi(\lambda_n,\rho)e^{-(\lambda_n^4/\alpha)t}\right\}, \tag{4.1}$$

where

$$\psi(\lambda_n,\rho) = \frac{[I_0(\lambda_n)J_1(\lambda_n)-J_0(\lambda_n)I_1(\lambda_n)][I_0(\lambda_n)J_0(\lambda_n\rho)-J_0(\lambda_n)I_0(\lambda_n\rho)]}{J_0(\lambda_n)I_0(\lambda_n)\{\lambda_n[I_0(\lambda_n)J_1(\lambda_n)-J_0(\lambda_n)I_1(\lambda_n)]-J_0(\lambda_n)I_0(\lambda_n)\}} \tag{4.2}$$

The eigenvalues λ_n are given as roots of the following transcendental equation

$$I_0(\lambda_n)J_1(\lambda_n)+J_0(\lambda_n)I_1(\lambda_n)-4\lambda_n J_0(\lambda_n)I_0(\lambda_n) = 0. \tag{4.3}$$

The first few values of λ_n are: $\lambda_1 = 2{\cdot}21$, $\lambda_2 = 5{\cdot}51$, $\lambda_3 = 8{\cdot}60\ldots$. The first term in expression (4.1) is the steady-state solution (homogeneous equation) whereas the second term represents the contribution of inertia forces. The series appearing in formula (4.1) is rapidly convergent and just one term provides an accuracy to within 1 per cent. By adding the second term, the accuracy increases to 0·001 per cent.

* Since t is dimensional time and coefficient α has dimension [sec] the exponent $(\lambda_n^4/\alpha)t$ is dimensionless.

Phase 1: $0 < t < \tau$

Let us assume that a transverse load p' is suddenly applied to the entire plate surface and after being kept constant during a certain interval of time τ is suddenly removed. In the first phase of motion the rate of deflection is given by the formula (4.1). The deflection of the plate can be obtained through a time-wise integration of (4.1) with initial condition (3.7) to give

$$w^{I}(\rho, t) = \frac{p'-p_0'}{256}\left\{(6\rho^4 - 28\rho^2 + 22)t - \alpha \sum_{n=1}^{\infty} \frac{256}{\lambda_n^9}\psi(\lambda_n, \rho)\left[1 - e^{-(\lambda_n^4/\alpha)t}\right]\right\}. \tag{4.4}$$

It is interesting to investigate the velocity of the plate at the end of the first phase because the motion and deflection during the second phase depend upon the value of kinetic energy attained at the instant $t = \tau$. For the constant value of applied impulse $I' = p'\tau = \text{const.}$, the plate velocity appears to be solely a function of τ. Let us introduce the dimensional impulse $I = (M_0/R^2)I'$ and dimensional velocity $V = BR^2 v$. In the limiting case of an ideal impulse ($\tau \to 0$), the velocity is found to be

$$\lim_{\tau \to 0} v(\rho, \tau) = \frac{I'}{\alpha} \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\psi(\lambda_n, \rho). \tag{4.5}$$

The series (4.5) appears to be an expansion of the constant unit function in terms of Bessel functions. Thus, the initial plate velocity in the case of an ideal impulse is constant along the radius and equals $v = I'/\alpha$ or $V = I/\mu$ which is the value expected from other considerations. However for $\tau > 0$ the velocity is no longer constant and depend upon all constants of the material.

Phase 2: $\tau < t < t_f$

After the load has been removed, the plate continues to deform until the kinetic energy input is dissipated in plastic work. The plate comes to rest at the instant t_f the value of which can be computed from the equation $v^{II}(\rho, t_f) = 0$. To determine the velocity and deflection of the plate, the requirement of continuity of $v(\rho, t)$ between the first and second phases of motion should be used

$$\begin{aligned} v^{II}(\rho, t) = & -\frac{p_0'}{256}(6\rho^4 - 28\rho^2 + 22) \\ & + \sum_{n=1}^{\infty} \frac{256}{\lambda_n^5}\left[\frac{p'}{256} - \frac{p'-p_0'}{256}e^{-(\lambda_n^4/\alpha)\tau}\right]\psi(\lambda_n, \rho)e^{-(\lambda_n^4/\alpha)(t-\tau)}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} w^{II}(\rho, t) = & \frac{p'\tau - p_0' t}{256}(6\rho^4 - 28\rho^2 + 22) + \\ & - \alpha \sum_{n=1}^{\infty} \frac{256}{\lambda_n^9}\psi(\lambda_n, \rho)\left[\frac{p'}{256}e^{-(\lambda_n^4/\alpha)(t-\tau)} - \frac{p'-p_0'}{256}e^{-(\lambda_n^4/\alpha)t} - \frac{p_0'}{256}\right]. \end{aligned} \tag{4.7}$$

Formulas (4.1), (4.4), (4.6) and (4.7) have been used to compute the central plate deflection and velocity as a function of time for the rectangular pressure pulse characterized by parameters $p' = 10$ and $\tau = 10^{-4}$ sec. These results are plotted on Fig. 2 as a full line. On the

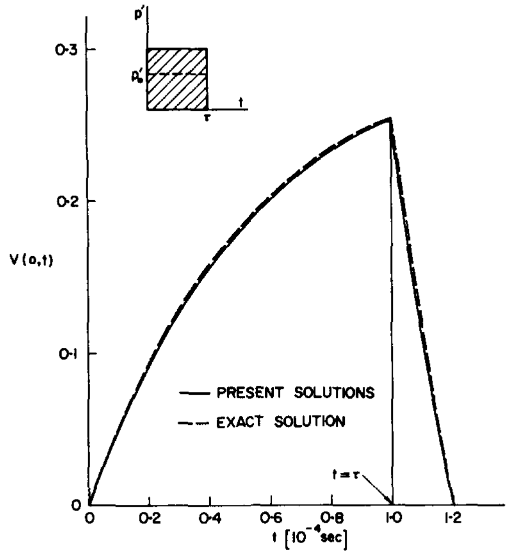

FIG. 2. Velocity of plate center vs. time for rectangular pulse shape. Results of present paper and exact numerical solution.

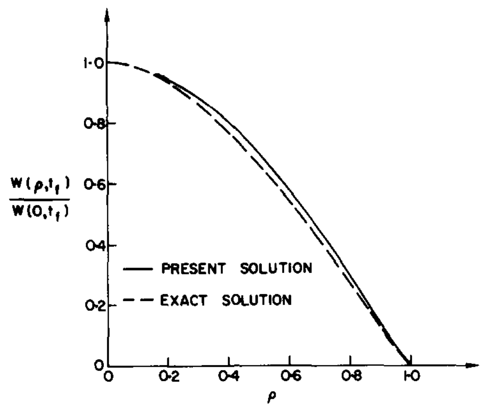

FIG. 3. Plate deflection curves, exact solution and present solution.

same figure, a broken line denotes the exact solution evaluated by means of numerical integration, see [3]. In both solutions the same value of parameters $\alpha = 10$ was assumed. Figure 3 presents the deflection profile at the end of phase 2 as given by the exact solution (broken line) and computed from (4.7), (solid line). It is interesting to note that the velocity fields presented substantially agree with the numerical computations of Eason [12] concerning rigid perfectly plastic plates with the Huber–Mises yield condition.

Results presented in the last three figures constitute a proof that the simplifying assumption concerning proportional loading leads to an excellent approximation of the exact solution.

5. IMPULSIVE LOADING, FINAL DEFLECTIONS OF PLATE, COMPARISON WITH EXPERIMENTS, DISCUSSION

The problem consists now in solving the nonhomogeneous differential equation (3.5) with boundary conditions (3.6) and initial condition (3.8). In this case $p' \equiv 0$. Following the method of eigenvalue expansion, the solution is found to be

$$v(\rho, t) = \frac{p'_0}{256}\left\{\sum_{n=1}^{\infty} \frac{256}{\lambda_n^5}\psi(\lambda_n, \rho)e^{-(\lambda_n^4/\alpha)t} + \sum_{n=1}^{\infty} \frac{256I'}{p'_0\alpha \cdot \lambda_n}\psi(\lambda_n, \rho)e^{-(\lambda_n^4/\alpha)t} - (6\rho^4 - 28\rho^2 + 22)\right\}. \tag{5.1}$$

The above solution satisfies the initial condition $v(\rho, 0) = I'/\alpha$. The first series appearing in (5.1) is rapidly convergent. The second series, however, converges very slowly, and is of the same structure as that in formula (4.5). A time integration of (5.1) provides an expression for the plate deflection

$$w(\rho, t) = \frac{p'_0}{256}\left\{\alpha \sum_{n=1}^{\infty} \frac{256}{\lambda_n^9}\psi(\lambda_n, \rho)(1 - e^{-(\lambda_n^4/\alpha)t}) + \frac{I'}{p'_0}\sum_{n=1}^{\infty} \frac{256}{\lambda_n^5}\psi(\lambda_n, \rho)(1 - e^{-(\lambda_n^4/\alpha)t}) - (6\rho^4 - 28\rho^2 + 22)t\right\} \tag{5.2}$$

Now both series appearing in (5.2) converge rapidly, so with an error of 1 per cent the approximate solution has the form

$$w(\rho, t) = \frac{p'_0}{256}(6\rho^4 - 28\rho^2 + 22)\left\{(1 - e^{-(\lambda_1^4/\alpha)t})\left(\frac{\alpha}{\lambda_1^4} + \frac{I'}{p'_0}\right) - t\right\} \tag{5.3}$$

The entire plate comes to rest at the instant $t = t_f$ when the velocity of all its points become zero, $v(\rho, t_f) = 0$ or when the deflections attain their maximum, $d/dt[w(\rho, t)|_{t=t_f}] = 0$. From both conditions we find that

$$t_f^{(n)} = \frac{3}{2}\frac{\alpha}{\lambda_n^4}\log_e\left(1 + \frac{2}{3}\frac{\lambda_n^4 I'}{3p'_0\alpha}\right) \tag{5.4}$$

Using terminology of the corresponding problem of elastic vibration of plates, we observe that different modes of "vibrations" $\psi(\rho, \lambda_n)$, $n = 1, 2, 3 \ldots$ vanish at different times $t_f^{(n)}$. Therefore, the plate would never come to rest at all its points simultaneously but, as pointed out, the first mode $\psi(\rho, \lambda_1)$ gives a good approximation to the solution. Thus, the approximate value $t_f = t_f^{(1)}$ will indicate when the first mode vanishes or when the deflections given by equation (5.3) attain their maximum. It is interesting to note that in the limiting case $\gamma \to \infty$ the duration time of the deformation process equals $t_f = I'/p'_0$ which in the case $p'_0 = 6$ coincides with an exact value known from the rigid, perfectly plastic analysis, Hopkins and Prager [5]. A slightly smaller value t_f is predicted by the present theory if $p'_0 = 6{\cdot}51$.

Substitution of (5.4) into (5.3) yields a satisfactory formula for the finite deflection of rigid, viscoplastic plates

$$w(\rho, t_f) = \frac{p'_0}{256}(6\rho^4 - 28\rho^2 + 22)\left\{\frac{I'}{p'_0} - \frac{3}{2}\frac{\alpha}{\lambda_1^4}\log\left(1 + \frac{2}{3}\frac{\lambda_1^4 I'}{p'_0\alpha}\right)\right\}. \tag{5.5}$$

In particular, the dimensional permanent central plate deflection δ is given by

$$\delta = 0\cdot97\frac{I^2R^2}{8\mu M_0}\left\{2\beta-2\beta^2\log\left(1+\frac{1}{\beta}\right)\right\}, \tag{5.6}$$

where

$$1/\beta = \frac{2}{3}\frac{\lambda_1^4 I'}{p_0'\alpha}.$$

If $\gamma \to \infty$, the limit value for the brackets in formula (5.6) is unity. In that case the present solution coincides with predictions based upon rigid, perfectly plastic theory. Similarly, for $\gamma \to 0$ the plate deflection is zero everywhere which corresponds to entirely rigid behavior. For all intermediate values of γ the strain rate diminishes the finite plate deflections relatively to the results of rigid, perfectly plastic analysis. The difference between both solutions as a function of applied impulse and viscosity is shown in Fig. 4. This difference

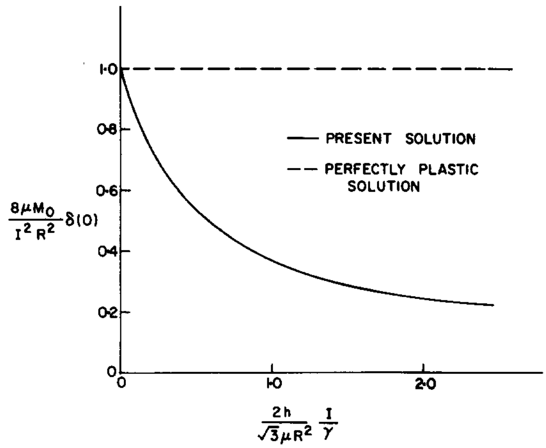

FIG. 4. Dimensionless central plate deflection. Difference between viscoplastic and perfectly plastic solution as a function of viscosity constant γ and value of applied impulse I.

increases with increasing value of the impulse I and decreasing value of the viscosity γ.

Formulas (5.6) have been used to compare viscoplastic behavior of impulsively loaded circular plates with results of recent experiments by Florence [4]. The tests reported in this paper concern thin, simply supported plates characterized by the following mechanical and geometrical parameters.

TABLE 1

Material	σ_0 (lb/in^2)	d (lb-sec^2/in^4)	$2h$ (in.)	R (in.)
C. R. Steel 1018	$7{\cdot}9\times10^4$	$7{\cdot}32\times10^{-4}$	0·241	4

The information available in the literature concerning the strain rate sensitivity of the material used is not consistent. Therefore curves for several values of γ have been shown on Fig. 5 (full lines). Experimental results are denoted by circles. An appropriate fit of

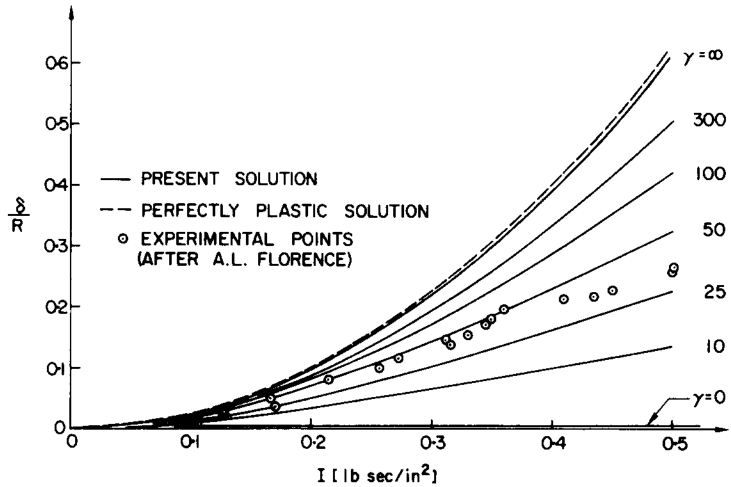

Fig. 5. Permanent central plate deflection δ/R vs. applied impulse I. Curves for several values of viscosity constant compared with experimental results (after Florence [4]) and with solution for perfectly plastic plates.

experimental points gives the curve corresponding to $\gamma = 50$. The present solutions indicate that, by including the strain rate effect even in a simple form, a considerable improvement in the description of dynamic response of plastic plates can be achieved. At the same time, it is seen from Fig. 5 that above a certain value of I the effect of membrane forces becomes more pronounced, as indicated by Florence [4].

The plot of final plate deflection against radius is shown on Fig. 6. The full line relates to the present solution and the broken line denotes the known solution of Wang [6]. Experimental points have been taken from the already mentioned paper. The present theory does not involve a discontinuity in slope at the plate center due to the smooth yield condition and therefore provides a better fit of the experimental plate profile in the central region of the plate.

For proper interpretation of discrepancies between the theory and experiments the assumption concerning the ideal impulse should be abandoned since the duration time of the acting pressure is always finite. Hence, the velocity at the end of the first phase will not be constant along the radius, as assumed in the present theory, but there are regions near the supported edges where the velocity falls from the value I/μ to zero. Even if this region is very narrow, the change of initial kinetic energy will be appreciable according to

$$E_k = \pi\mu \int_0^1 v^2(\rho, \tau)\rho \, d\rho. \qquad (5.7)$$

For the constant initial velocity distribution equation (5.7) yields $E_k = I^2\pi/2\mu$. This value is overestimated by all existing theories, including the present approach, and this influences of course the final plate deflection. Measurements of the initial velocity distribution or time dependence of pressure are necessary to introduce a correction in the present theory.

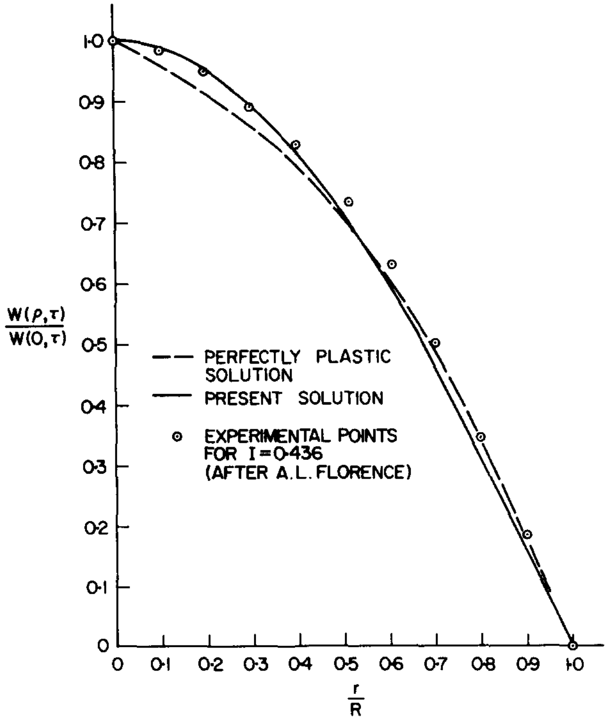

FIG. 6. Final deformation profile predicted by viscoplastic and perfectly plastic solution compared with experimental data.

6. CONCLUDING REMARKS

Equations describing the dynamic behavior of thin rigid, viscoplastic plates with Huber–Mises yield condition are essentially nonlinear. In the present paper, a hypothesis concerning proportional loading has been introduced to linearize the problem and derive a solution in an analytical form. Since the equations obtained are of the same structure as those describing the elastic vibration of thin plates, many of the methods of solutions developed in the theory of elasticity can be applied in the case of viscoplastic problems. For example, the dynamics of simply supported square and rectangular plates is described by an equation similar to (3.5) but the Laplace operator ∇^2 is now defined in Cartesian coordinates. The solution of this problem involves trigonometric functions and the eigenvalues are given in an explicit form. In the limiting case ($\gamma \to \infty$) a full transition to the known solution of Cox and Morland [13] for perfectly plastic materials is obtained. For the details of this solution, see [14]. The results concerning simply supported plates can be easily extended to other conditions of plate support, in particular the clamped edge conditions. This will be a subject of a separate paper.

Although the constitutive equations used throughout the paper have been assumed in the simple form of a linear law between stresses and strain rates, the resulting solution provides a much better fit of experimental data than a classical solution based upon perfectly plastic material. For the remaining discrepancy between theory and experiments two effects are responsible. These are the neglect of membrane forces which come into play for large values of applied pressure and also overestimation of the initial kinetic energy. While the latter source of discrepancy can be easily eliminated by abandoning the assumption concerning ideal impulse, consideration of membrane forces would lead again to nonlinear problems, and this remains still an open question.

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Résumé—La réaction de plaques métalliques circulaires fines simplement supportées à une charge impulsive est investiguée. Les équations constitutives employées sont basées sur la condition de la plasticité Huber–Mises et tiennent compte de l'influence de la vitesse de la deformation. L'hypothèse concernant le chargement proportionnel pour un matériel viscoplastique a été employée pour linéariser le système des équations gouvernantes. Des formules explicites ont été obtenues pour la vélocité de la plaque et la déflection plastique permanente sous diverses conditions de chargement de plaque. Une comparaison avec des données expérimentales récentes et les solutions pour les plaques parfaitement plastiques connues est présentée. Il est constaté que la forme de la plaque déformée prédite par la théorie se conforme toujours à la déflection observée. Par un choix approprié des constantes de matériel une bonne concordance peut être réussie, entre la théorie et des expériences, à un certain intervalle de l'impulsion appliquée.

Zusammenfassung—Es werden frei-drehbar gestütste dünne Metallkreisplatten unter der dynamischen Belastung untersucht. Die hier angenommenen Stoffgleichungen gründen sich auf der Huber–von Mises'chen Plastizitätsbedingung; der Einfluss der Verformungsgeschwindigkeit wird auch berucksichligt. Durch die Annahme der Hypothese der proportionellen Belastung der visco-plastischen Stoffe wurde das Gleichungssystem linearisient. Bei verschiedenen Arten der dynamischen Belastung erhielt man die Formeln für die Plattengeschwindigkeit und die plastischen Durchbiegungen. Die Vergleichung der bekommenen Ergebnisse mit den bekannten Lösungen für die ideal plastischen Platten wurde durchgeführt. Die auch vorgestellte Vergleichung mit den Versuchsergebnissen wies eine gute Übereinstimmung der Theorie mit dem Experimenten auf. Diese Übereinstummung kann durch die entsprechende Wahl der Stoffkonstanten in einem grossen Bereich der Impulse erzielt werden.

Абстракт—Исследуется паведение тонких, свободноопёртых круговых металлических пластин под действиет импульсного нагружения. Применённые определяющие уравнения основаны на условии текучести Губер-Мизеса и учитывают зффект чувствительности материала кскорости деформации. Для линеализирования системы уравнений применяется гипотеза, о пропорциональном нагружении для вязко-пластичного материала. В явном виде получены формулы для скорости пластины и для пластического прогиба при различных условиях нагружении пластины. Дается сравнение с последними зкспериментальными данными и с известным идеально-пластическим решением. Найдено, что форма деформированной пластины, предсказанная теорией всегда совиодаеги с опытными данными. В некоторагх интерьалах ирилагаемого имиулбса, при соответствующем выборе постоянных материала, может быть достигнута большая степень согласованности между теорией и зкспериментами.